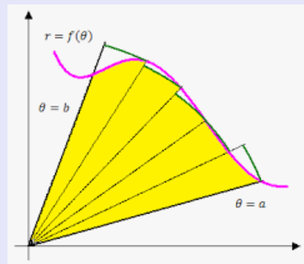
Calculus II

Lecture 22



Feb 19-8:47 AM

Class QZ 19

Find the $\boxed{\text{exact}}$ arc length of the polar curve $\boxed{r = \theta^2}$ for $\boxed{0 \le \theta \le \sqrt{5}}$ by using

$$L = \int_\alpha^\beta \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta.$$

$r^2 = (\theta^2)^2 = \theta^4$

$\left(\frac{dr}{d\theta}\right)^2 = (2\theta)^2 = 4\theta^2$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta$$

$\theta = 0 \to u = 4$ $\quad u = \theta^2 + 4$

$\theta = \sqrt{5} \to u = 9$ $\quad du = 2\theta\, d\theta$

$\frac{du}{2} = \theta\, d\theta$

$$= \int_4^9 \sqrt{u}\, \frac{du}{2}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{u^{3/2}}{3/\cancel{2}}\Big|_4^9 = \frac{1}{3} u\sqrt{u}\Big|_4^9$$

$$= \frac{1}{3}\left[9\sqrt{9} - 4\sqrt{4}\right] = \frac{1}{3}\left[27 - 8\right] = \boxed{\frac{19}{3}}$$

Jul 17-7:20 AM

Test for Convergence or Divergence:

1) $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

because of Power of n → Root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n+1)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{2n+1}{n^2} = 0 < 1$$

Converges by root test.

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2) $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$

Because of ! , we do ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{2^{n+1} \cdot (n+1)!}{(n+3)!}}{\frac{2^n \cdot n!}{(n+2)!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^{n+1}} \cdot \cancel{(n+1)!} \cdot \cancel{(n+2)!}}{\cancel{2^n} \cdot \cancel{n!} \cdot \cancel{(n+3)!}} = \lim_{n\to\infty} \frac{2(n+1)}{(n+3)}$$

$= 2 > 1$

Diverges by Ratio Test

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3) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

Since n^2 is in derivative of $n^3 \Rightarrow$ try Integral Test

$\sum_{n=1}^{\infty} n^2 e^{-n^3}$ $f(x) = x^2 e^{-x^3}$

1) Positive terms

2) Cont. 3) Decreasing

$\int_1^{\infty} x^2 e^{-x^3} dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3} dx = \lim_{t\to\infty} \left[\frac{-1}{3} e^{-x^3}\right]_1^t$

$\int x^2 e^{-x^3} dx = \int e^{-u} \frac{du}{3} = \frac{1}{3} \frac{e^{-u}}{-1} = \frac{-1}{3} e^{-u}$

$u = x^3$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$= \lim_{t\to\infty} \left[\frac{-1}{3}\left(\cancel{e^{-t^3}}^{\to 0} - e^{-1}\right)\right]$

$= \frac{1}{3e}$

Improper integral Converges

Our Series Converges by I.T.

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4) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

what do you recall from trig.?

$-1 \leq \sin\alpha \leq 1$

Since $\sin 2n \leq 1$, then $\frac{\sin 2n}{1+2^n} \leq \frac{1}{1+2^n}$

Try Comparison or limit Comparison test.

Let's compare $\frac{1}{1+2^n}$ with $\frac{1}{2^n}$

$\frac{\sin 2n}{1+2^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ Converges by I.G.S. $r = \frac{1}{2}$

by Comparison test

$\sum \frac{\sin 2n}{1+2^n}$ Converges

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5) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

Since we have $(-1)^{n-1}$, use Alternating Series test.

$a_n = \dfrac{1}{\sqrt{n}-1}$

1) $a_{n+1} < a_n$ ✓

2) $\displaystyle\lim_{n\to\infty} a_n = 0$ ✓

Converges by A.S.T.

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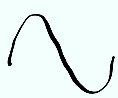
6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sin n}$

Divergence Test

$\displaystyle\lim_{n\to\infty} \frac{1}{2+\sin n}$ does not exist.



Our Series diverges by Divergence Test.

7) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

Try Ratio Test

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2+1}{5^{n+1}}}{\frac{n^2+1}{5^n}}$$

$$=\lim_{n\to\infty} \frac{\cancel{5^n}(n^2+2n+2)}{5\,\cancel{5^{n+1}}(n^2+1)}$$

$$= \frac{1}{5}\lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} = \frac{1}{5}\cdot 1 = \frac{1}{5} < 1$$

It converges by Ratio Test.

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8) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ Try Root Test, Expand Factorial.

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^{4n}}} = \lim_{n\to\infty} \frac{n!}{n^4}$$

Hint: $n! = n(n-1)(n-2)(n-3)\cdot \ldots \cdot 3\cdot 2\cdot 1$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)\cdot \ldots \cdot 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdot n}$$

$$= \lim_{n\to\infty} \left[\cancel{\frac{n}{n}}^{\to 1} \cdot \cancel{\frac{n-1}{n}}^{\to 1} \cdot \cancel{\frac{n-2}{n}}^{\to 1} \cdot \cancel{\frac{n-3}{n}}^{\to 1} \cdot (n-4)\ldots 3\cdot 2\cdot 1 \right]$$

$= \infty > 1$ It diverges by Root Test

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9) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\cos^2 n)}$ Hint: what do You know from trig. about $\cos\alpha$?

Let's compare $\dfrac{1}{n(1+\cos^2 n)}$ with $\dfrac{1}{2n}$

$-1 \leq \cos\alpha \leq 1$

$0 \leq \cos^2\alpha \leq 1$

$1 \leq 1+\cos^2\alpha \leq 2$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ Diverges

1) I.T.

2) P-Series

3) Harmonic Series

using Comparison Test

It diverges.

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10) $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1) = \sum_{n=1}^{\infty} \left[2^{\frac{1}{n}} - 1\right]$

do limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{1/n}-1}{1/n} = \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{2^{\frac{1}{n}} \cdot \ln 2 \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = 2^0 \cdot \ln 2 = \boxed{\ln 2}$$

our Series diverges by L.C.T.

Jul 18-9:22 AM

Test these two Series

1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

P-Series, $P=2>1$

Converges by P-Series test

2) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^2}$

Alternating Series

$a_n = \frac{1}{n^2}$

$a_{n+1} < a_n$

$\lim_{n\to\infty} a_n = 0$

Converges by A.S.T.

$\sum_{n=1}^{\infty} a_n$ is absolutely Convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^2}$ Converges abs. since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

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$$\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$$

Converges by G.S.T.

$$\underbrace{\sum_{n=1}^{\infty} \left| \frac{\sin 2n}{1+2^n} \right|}_{} \le \sum_{n=1}^{\infty} \frac{1}{1+2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Converges by Comparison Test

$\to$ Converges Abs.

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Power Series

It is a Series that Contains a Variable other than n.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$$

Try Ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}\ \cancel{x^{n+1}}}{(n+1)^2}}{\frac{\cancel{(-1)^n}\ \cancel{x^n}}{n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-1 \cdot x \cdot n^2}{(n+1)^2} \right| = |x| \left\{ \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \right\} = |x|$$

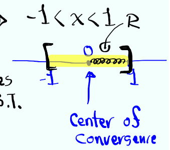
$\to 1$

Ratio test if limit $< 1 \to$ Converges

$|x| < 1 \to -1 < x < 1$

at $x = 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$ Converges by A.S.T.



at $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{\left[(-1)^2\right]^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Converges by P-Series

Interval of Convergence $[-1, 1]$

Center of Convergence 0

Radius of Convergence 1

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Consider the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$$

Ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)\cdot 3^{n+1}}}{\frac{x^n}{n\,3^n}}\right| = \lim_{n\to\infty}\left|\frac{n\,\cancel{3^n}\,\cancel{x^{n+1}}}{(n+1)\cdot \cancel{3^{n+1}}\cdot \cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x\,n}{3(n+1)}\right| = \left|\frac{x}{3}\right|\underbrace{\left|\lim_{n\to\infty}\frac{n}{n+1}\right|}_{\to 1} = \left|\frac{x}{3}\right|$$

Power Series Converges if $\underbrace{\text{limit of Ratio}}<1$

$$\left|\frac{x}{3}\right|<1$$

$$\vdots$$

$$-3<x<3$$

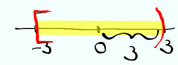
If $x=3$

$$\sum \frac{x^n}{n\cdot 3^n} = \sum\frac{3^n}{n\cdot 3^n} = \sum\frac{1}{n}$$ Diverges by H.S.

If $x=-3$

$$\frac{(-3)^n}{3^n} = \frac{(-1)^n\cdot 3^n}{3^n} = (-1)^n$$

$$\sum\frac{x^n}{n3^n} = \sum\frac{(-3)^n}{n\,3^n} = \sum\frac{(-1)^n}{n}$$ Converges by A.S.T.



Interval of Convergence $[-3,3)$

Center of " 0

Radius of " 3

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Consider the Power Series below

$$\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}}$$

Try Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\frac{(2x-1)^n}{5^n\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}\,(2x-1)}{5\,\sqrt{n+1}}\right| = \frac{|2x-1|}{5}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|2x-1|}{5}$$

Power Series Converges when $\frac{|2x-1|}{5}<1$

$$|2x-1|<5$$

$$-5<2x-1<5$$

$$-4<2x<6$$

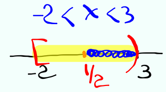
$$-2<x<3$$

$x=3$

$$\sum\frac{(2\cdot3-1)^n}{5^n\cdot\sqrt{n}} = \sum\frac{1}{\sqrt{n}}$$ Diverges by P-Series, $P=\frac{1}{2}$

$x=-2$

$$\sum\frac{(2\cdot -2-1)^n}{5^n\sqrt{n}} = \sum\frac{(-1)^n}{\sqrt{n}}$$ Converges by A.S.T.



Interval of Convergence $[-2,3)$

Center of Convergence $\frac{1}{2}$

Radius of Conv. $\frac{5}{2}$

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Class QZ 20 $\boxed{\text{Open Notes}}$

Find $\boxed{\text{exact Ans}}$ for surface area when the length of the curve given by $x=3t^4$, $y=4t^3$ for $0 \le t \le \sqrt{3}$ is rotated about x-axis.

$$S=\int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

$\frac{dx}{dt}=12t^3$

$\frac{dy}{dt}=12t^2$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=144t^6+144t^4=144t^4(t^2+1)$$

$$S=\int_0^{\sqrt{3}} 2\pi\cdot 4t^3\cdot\sqrt{144t^4(t^2+1)}\,dt=\int_0^{\sqrt{3}} 2\pi\cdot 4t^3\cdot 12t^2\sqrt{t^2+1}\,dt$$

$$=96\pi\int_0^{\sqrt{3}} t^5\sqrt{t^2+1}\,dt \Rightarrow$$

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$$=96\pi\int_0^{\sqrt{3}} t^5\sqrt{t^2+1}\,dt$$

$u=t^2+1$ $\quad du=2t\,dt$

$t^2=u-1$ $\quad \frac{du}{2}=t\,dt$

$t=0 \to u=0^2+1=1$

$t=\sqrt{3} \to u=(\sqrt{3})^2+1=4$

$$=96\pi\int_0^{\sqrt{3}} t^4\sqrt{t^2+1}\cdot t\,dt=96\pi\int_1^4 (u-1)^2\sqrt{u}\,\frac{du}{2}$$

$$=48\pi\int_1^4 (u^2-2u+1)\sqrt{u}\,du=48\pi\left[\frac{u^{7/2}}{7/2}-\frac{2u^{5/2}}{5/2}+\frac{u^{3/2}}{3/2}\right]\Bigg|_1^4$$

$$=48\pi\left[\frac{2}{7}u^3\sqrt{u}-\frac{4}{5}u^2\sqrt{u}+\frac{2}{3}u\sqrt{u}\right]\Bigg|_1^4$$

$$=48\pi\left[\left(\frac{2\cdot 4^3\cdot\sqrt{4}}{7}-\frac{4\cdot 4^2\sqrt{4}}{5}+\frac{2\cdot 4\sqrt{4}}{3}\right)-\left(\frac{2}{7}-\frac{4}{5}+\frac{2}{3}\right)\right]=$$

$$=48\pi\left[\frac{256}{7}-\frac{128}{5}+\frac{16}{3}-\frac{2}{7}+\frac{4}{5}-\frac{2}{3}\right]=$$

$$=48\pi\left[\frac{254}{7}-\frac{124}{5}+\frac{14}{3}\right]=48\pi\cdot\frac{1696}{105}=\boxed{\frac{27136\pi}{35}}$$

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